ALGEBRA / Series

FACTORIALS

You should already know that a factorial number is denoted with ! so 4! means factorial 4.

$4! = (4)(3)(2)(1) = 24$ $n!=(n)(n-1)(n-2)(n-3).....1$ **Without proof, 0! Is always taken as 1**

SUMMATION of a SERIES

Many expressions can be represented by a series of numbers added together.

This might be a series of numbers like $x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+.........\frac{1}{n}$

It might be a power series of x like $y = x^0 + x^1 + x^2 + x^3 + ...x^n$

In general a series may be written as $x = u_1 + u_2 + ...u_n$ where u is the term in the series.

Consider the following calculation. $x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+.........\frac{1}{n}$

The value of x is the sum of a series of fractions with the numerators forming a descending series of integers (whole numbers). We could write this more simply as: $x = \sum_{n=1}^{n=n}\frac{1}{n}$ or $\sum_{1}^{n}\frac{1}{n}$

The symbol $\sum$ is a capital letter Sigma and means the 'sum of '
The limits of the variable between the summation takes place is shown.

WORKED EXAMPLE No. 1

Write out the series represented by $\sum_{1}^{n}\frac{4n^2x^n}{2n^2+n-2}$ for the first 3 terms.

SOLUTION

$n = 1 \quad \frac{4n^2}{2n^2+n-2} = \frac{4}{2+1-2} = 4$ $\qquad n = 2 \quad \frac{4n^2}{2n^2+n-2} = \frac{16}{8+4-2} = \frac{16}{10}$

$n = 3 \quad \frac{4n^2}{2n^2+n-2} = \frac{36}{18+3-2} = \frac{36}{19}$

hence $\sum_{1}^{n}\frac{4n^2x^n}{2n^2+n-2} = 4x + \frac{16x^2}{10} + \frac{36x^3}{19} + ...\frac{4n^2x^n}{2n^2+n-2} + ...$

WORKED EXAMPLE No. 2

Write out the series represented by $\sum_{1}^{n}(-1)^{n-1}\frac{x^n}{n!}$ for the first 4 terms.

SOLUTION

$n = 1 \quad (-1)^{n-1}\frac{x^n}{n!} = (-1)^0\frac{x^1}{1} = x$ $\qquad n = 2 \quad (-1)^{n-1}\frac{x^n}{n!} = (-1)^1\frac{x^2}{2} = -\frac{x^2}{2}$

$n = 3 \quad (-1)^{n-1}\frac{x^n}{n!} = (-1)^2\frac{x^3}{(3)(2)} = \frac{x^3}{6}$ $\qquad n = 4 \quad (-1)^{n-1}\frac{x^n}{n!} = (-1)^3\frac{x^4}{(4)(3)(2)} = -\frac{x^4}{24}$

Hence $\sum_{1}^{n}(-1)^{n-1}\frac{x^n}{n!} = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + ...(-1)^{n-1}\frac{x^n}{n!} + ...$

Factorials & Series

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LIMITING VALUES

Consider the following equation. $x = \frac{2n^2 + n - 4}{7n^2 + 6n - 20}$ Suppose we wish to know if this has a value when $n = \infty$. A simple way to find out is to make use of the fact that $1/\infty = 0$

If we rearrange the equation by dividing through by the highest order of n (n^2 in this case) we get

$$x = \frac{2 + 1/n - 4/n^2}{7 + 6/n - 20/n^2}$$

Now put $n = \infty$ $\quad x = \frac{2 + 0 - 0}{7 + 0 - 0} = \frac{2}{7}$ so there is a limiting value when $n = \infty$

We write this as $\quad \mathcal{L}_{n \to \infty} x = \frac{5}{7}$

WORKED EXAMPLE No. 3

Find the limiting value of the following expressions when $n \to \infty$

i) $\frac{4n^2}{2n^2 + n - 2}$ ii) $\frac{n}{(3n-2)(5n+7)}$ iii) $\frac{n^3 + 5n^2 + 5}{n^2}$

SOLUTION

i) $\frac{4n^2}{2n^2 + n - 2} = \frac{4}{2 + 1/n - 2/n^2} = 2$ ii) $\frac{n}{(3n-2)(5n+7)} = \frac{1/n}{(3 - 2/n)(5 + 7/n)} = \frac{0}{15} = 0$

iii) $\frac{n^3 + 5n^2 + 5}{n^2} = \frac{1 + 5/n + 5/n^3}{1/n} = \frac{1 + 0 + 0}{0} = \frac{1}{0} = \infty$

SELF ASSESSMENT EXERCISE No.1

1. Write out the series represented by $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$ for the first 3 terms.

Answer $\quad x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \ldots\ldots(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$

2. Find the limiting value of the following expressions when $n \to \infty$

i. $5 + \frac{2}{n^2}$ ii. $\frac{5n - 2}{2n + 1}$ iii. $\frac{(n+2)(n+3)}{n(n+1)(n+4)}$

Answers 5, 2.5 and 0

CONVERGENCE and DIVERGENCE

In the series $x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots\ldots\frac{1}{n}$ we might think that since each term is smaller than the one before it, then the value of x would tend to converge on some figure as we add more and more terms. In fact the value of x will go on getting bigger so in the limit as $n\to\infty$ the value of x will also tend to infinity. This series has no limiting value. We must be very careful dealing with series because the value of each term may get bigger and bigger (divergence) or it might get smaller and smaller (convergence) and if it converges there is a limiting value.

In order to find the value of a series when $n = \infty$ we write the series in the form:

$$x = u_1 + u_2 + u_3 + \ldots\ldots u_n \text{ or } x = \sum_{n=1}^{n=\infty} u_n$$

If the term at $n = \infty$ is **not zero** then it seems likely that series has no limiting value and is divergent. If $\underset{n\to\infty}{\mathscr{L}t}\ u_n = 0$ the series might converge to a limiting value but this is not certain.

For example consider again the following series i. $x = \sum_{n=1}^{n=\infty}\left(\frac{1}{n}\right)$ The series is $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\frac{1}{n}$

$\underset{n\to\infty}{\mathscr{L}t}\left(\frac{1}{n}\right) = 0$ It was shown earlier that this series in divergent so having a zero value does not prove the series is convergent.

WORKED EXAMPLE No. 4

Determine if the following series is divergent. $x = \sum_{n=1}^{n=\infty}\left(\frac{3n+2}{10n+1}\right)$

SOLUTION

The series is $\frac{5}{11}+\frac{8}{21}+\frac{11}{31}+\ldots\ldots\frac{3n+2}{10n+1}$

$\underset{n\to\infty}{\mathscr{L}t}\left(\frac{3n+2}{10n+1}\right) = \left(\frac{3+2/n}{10+1/n}\right) = \left(\frac{3}{10}\right)$ and since this is not zero the series is divergent.

D'ALEMBERT'S RATIO

For any series $x = \sum_{n=1}^{n=\infty} u_n$

$\underset{n\to\infty}{\mathscr{L}t}\left(\frac{u_{n+1}}{u_n}\right) < 1$ the series converges.

$\underset{n\to\infty}{\mathscr{L}t}\left(\frac{u_{n+1}}{u_n}\right) > 1$ the series diverges.

This does not tell us what happens if the result is unity.

WORKED EXAMPLE No. 5

Determine if the following series is divergent. $x = \sum_{n=1}^{n=\infty}\left(\frac{2n-1}{2^{n-1}}\right)$

SOLUTION

The series is $x = 1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \ldots\ldots\ldots\ldots \frac{2n-1}{2^{n-1}}$

We note that $u_n = \frac{2n-1}{2^{n-1}}$ and $u_{n+1} = \frac{2(n+1)-1}{2^{n+1-1}} = \frac{2n+1}{2^n}$

$$\frac{u_{n+1}}{u^n} = \frac{(2n+1)(2^{n-1})}{2^n(2n-1)} = \frac{2^n(2n+1)(2^n)2^{-1}}{2^n(2n-1)} = \frac{(2n+1)}{2(2n-1)} = \frac{(2+1/n)}{2(2-1/n)}$$

$\mathop{\mathscr{L}t}\limits_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{2+0}{2(2+0)} = \frac{2}{4} = \frac{1}{2}$ This is less than 1 so the series is convergent

SELF ASSESSMENT EXERCISE No. 2

Find the limiting value of the following expressions when $n \to \infty$

1. Determine if the following are divergent or convergent.

i. $x = \sum_{n=1}^{n=\infty}\left(\frac{2^{n-1}}{n+9}\right)$ ii) $x = \sum_{n=1}^{n=\infty}\left(\frac{n}{n+1}\right)$

Answers

i. $\mathop{\mathscr{L}t}\limits_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 2$ hence divergent.

ii. $\mathop{\mathscr{L}t}\limits_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1$ hence indeterminate

but $\mathop{\mathscr{L}t}\limits_{n\to\infty} u_n = 1$ and since this is not zero it must be divergent.

ABSOLUTE CONVERGENCE

Without proof it can be shown that if we determine the sum of the modulli of each term in a series such that $S = |u_1| + |u_2| + |u_3| + ...|u_n| = \sum |u_n|$ and if this is convergent, then $\sum u_n$ is also convergent and has a definite value hence the use of the words absolute convergence.

It follows that if all the terms are positive anyway, then if the series is convergent it is absolutely convergent

WORKED EXAMPLE No. 6

Test the following series to see if it has absolute convergence. $1 - \frac{3}{2} + \frac{5}{2^2} - \frac{7}{2^3} + ...$

SOLUTION

$$\sum |u_n| = |1| + \left|-\frac{3}{2}\right| + \left|\frac{5}{2^2}\right| + \left|-\frac{7}{2^3}\right| + ... = 1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + ...\sum_{n=1}^{n=\infty}\left(\frac{2n-1}{2^{n-1}}\right)$$

This is now the same as example 5 whence

$$\mathscr{L}t_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{2+0}{2(2+0)} = \frac{2}{4} = \frac{1}{2}$$

This is less than 1 so the series is convergent and the series $1 - \frac{3}{2} + \frac{5}{2^2} - \frac{7}{2^3} + ...$ is absolutely convergent.

WORKED EXAMPLE No. 7

Determine if the following power series is convergent or divergent.

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!}....\frac{x^{n-1}}{(n-1)!}$$

SOLUTION

$$u_n = \frac{x^n}{n!} \qquad u_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{n!\,x^{n+1}}{x^n(n+1)!} = \frac{x}{n+1}$$

$$\mathscr{L}t_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{x}{\infty} = 0 \text{ for all values of x}$$

This is less than 1 so the series is absolutely convergent for all values of x.

SELF ASSESSMENT EXERCISE No. 3

Test the following series for convergence.

1. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...(-1)^{n-1}\frac{x^n}{n}$
2. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$

Answers

1. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|$ so if $|x|<1$ the series is absolutely convergent but if $|x|>1$ it is divergent.
2. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0$ for all values of x hence the series is absolutely convergent for all x.

PASCAL'S TRIANGLE

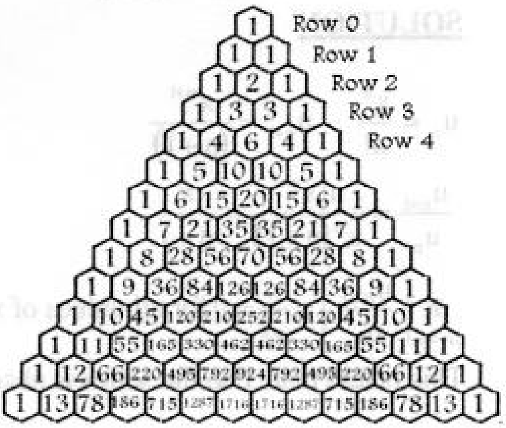
An important factorial expression is $^nC_r = \frac{n(n-1)(n-2)(n-3)......(n-\{r-1\})}{r!}$

The top line is the first r factors of n and the bottom line is factorial r

If we evaluated all the values of ^nC_r from r = 0 to r = n we would find the values are symmetrical. For example take the case n = 5

$^5C_0 = \frac{1}{1} = 1$ $^5C_1 = \frac{5}{1} = 5$ $^5C_2 = \frac{(5)(4)}{(2)(1)} = 10$ $^5C_3 = \frac{(5)(4)(3)}{(3)(2)(1)} = 10$ $^5C_4 = \frac{(5)(4)(3)(2)}{(4)(3)(2)(1)} = 5$

$^5C_5 = \frac{(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)} = 1$



Pascal's Triangle is made of rows as shown.
The n^{th} row is made of all the numbers ^nC_r for r = 0 to r = n
The zero row is $^0C_0 = 1$
The 1^{st} row is $^1C_0 = 1$ $^1C_1 = 1$
The 2^{nd} row is $^2C_0 = 1$ $^2C_1 = 2$ $^2C_2 = 1$
The 3^{rd} row is $^3C_0 = 1$ $^3C_1 = 3$ $^3C_2 = 3$ $^3C_3 = 1$