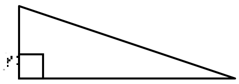
Pythagorean theorem and trigonometry:



We hereby remind you of the basic trigonometric ratios and use them to solve triangles and apply them to the resolution of forces. What we mean by solving triangles is to find their missing angles and/or sides.

Fundamental trigonometric ratios

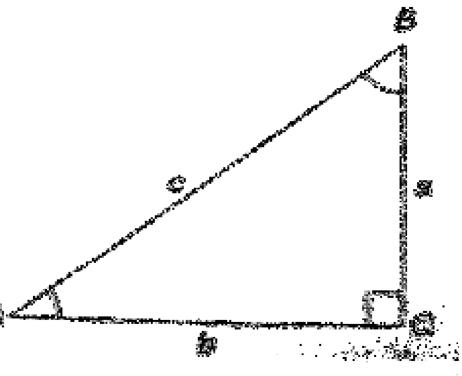
I am sure that in your previous studies you have met the fundamental trigonometric ratios. However, they are an essential part of this outcome and for those with this gap in their knowledge and to serve as a reminder to others, they are repeated here.

For any right-angled triangle (*Figure 4.14*),

1. $\dfrac{\text{The side opposite the angle}}{\text{the hypotenus}}$ is called the **sine** of the reference angle and is often abbreviated to *sin*. Therefore,

$$\sin A = \frac{opposite}{hypotenuse} = \frac{a}{c}.$$

Note that from now on we will use only the capital letter to represent angles, dropping the $\hat{A}$ (hat) sign above the letter. Also note that *lower case letters represents the sides of the triangle.*



$$\sin A = \frac{\text{opp.}}{\text{hyp.}} = \frac{a}{c} \quad \cos A = \frac{\text{adj.}}{\text{hyp.}} = \frac{b}{c} \quad \tan A = \frac{\text{opp.}}{\text{adj.}} = \frac{a}{b}$$

Note: 1. The hypotenuse is opposite the right angle

2. $\tan A = \dfrac{\sin A}{\cos A}$

Figure 4.14 The right-angled triangle

2. $\frac{\text{The side adjacent to the angle}}{\text{the hypotenuse}}$ is called the **cosine** of the reference angle and is often abbreviated to ***cos***. Therefore,

$$\cos A = \frac{adjacent}{hypotenuse} = \frac{b}{c}$$

3. $\frac{\text{The side opposite the angle}}{\text{the side adjacent to the angle}}$ is called the **tangent** of the angle and is often abbreviated to *tan*. Therefore,

$$\tan A = \frac{opposite}{adjacent} = \frac{a}{b}$$

These fundamental ratios are very important and should be remembered. One aid to memory is to use the acronym **SOHCAHTOA** (pronounced *sock-co-toe-ah*). Where the letters mean: sine, opposite, hypotenuse; cosine, adjacent, hypotenuse; tangent, opposite, adjacent. There are other aids to memory, stick with what you know!

One final very important point concerning these ratios, remember that the ***sine, cosine and tangent functions must be followed by the angle they refer to***. Thus, for example, *sin A*, makes sense, but *sin* on its own is nonsense. Remember that the *sin* and the *A*, ***cannot be separated!***

Solving right-angled triangles

So far we have used the trigonometric ratios: sine, cosine and tangent to find angles, given the 3 sides of a triangle. We can in fact solve any right-angled triangle, given any side and two angles using the trigonometric ratios, and where necessary, combining these ratios with Pythagoras.

Consider the triangle shown in *Figure 4.16*, with sides, *a*, *b*, *c* and angles, $\angle A$, $\angle B$ and $\angle C$.

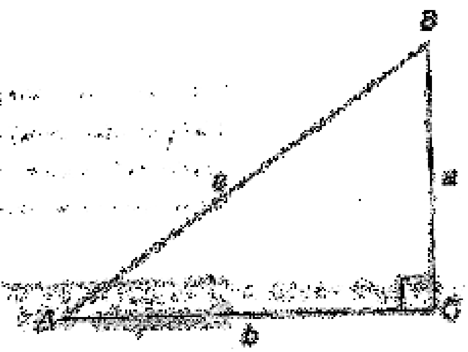


Figure 4.16 Labelled right-angled triangle

Then from the figures:

$$\sin\theta = \frac{a}{c}, \text{ therefore } a = c\sin\theta$$

$$\cos\theta = \frac{b}{c}, \text{ therefore } b = c\cos\theta$$

$$\tan\theta = \frac{a}{b}, \text{ therefore } a = b\tan\theta$$

Also, from Pythagoras: $c^2 = a^2 + b^2$

The sine and cosine rules

We now extend our knowledge to the solution of triangles, which are not right-angled. In order to do this we need to be armed with just two additional formulae. These are tabulated below for reference.

Sine rule	$\frac{a}{\sin a} = \frac{b}{\sin b} = \frac{c}{\sin c}$
Cosine rule	$a^2 = b^2 + c^2 - 2bc\cos A$
	$b^2 = a^2 + c^2 - 2ac\cos B$
	$c^2 = a^2 + b^2 - 2ab\cos C$

The above rules can only be used in specific circumstances.

For the general triangle ABC shown in *Figure 4.29*, with sides a, b, c and angles, $\angle A$, $\angle B$, $\angle C$. Then the ***sine rule may only be used when either***

- ***one side and any two angles are known***

or if

- ***two sides and an angle (not the angle between the sides) are known.***

The cosine rule may only be used when either

- ***three sides are known***

or

- ***two sides and the included angle are known.***

Note: When using the sine rule, the equality signs allow us to use any parts of the rule that may be of help. For example, if we have a

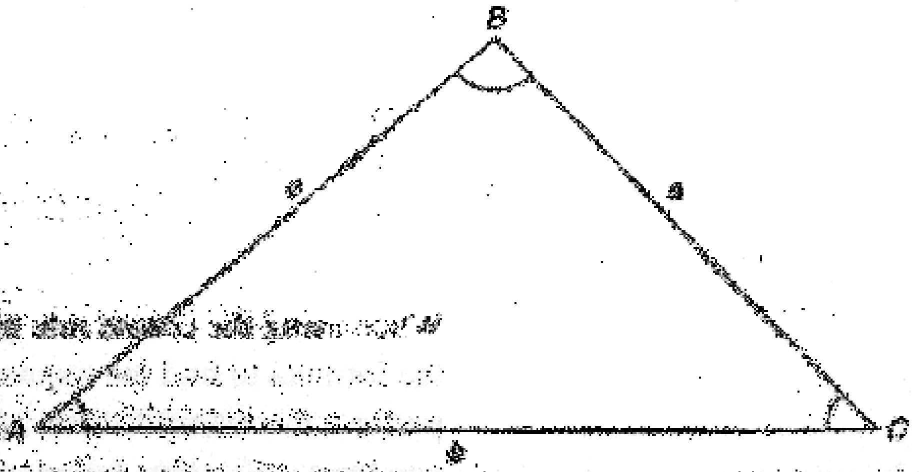


Figure 4.29 The general triangle

triangle to solve, for which we know the angles $\angle A$ and $\angle C$ and side a. We would first use the rule with the terms: $\frac{a}{\sin A} = \frac{c}{\sin C}$, to find side c.

Note: When using the cosine rule, the version chosen will also depend on the information given. For example, if you are given sides a, b and the included angle C, then the formula: $c^2 = a^2 + b^2 - 2ab\cos C$ would be selected to find the remaining side c.

Summary of

Trigonometric Formulae

$\cos^2 A + \sin^2 A = 1$ $\quad \sec^2 A - \tan^2 A = 1$ $\quad \mathrm{cosec}^2 A - \cot^2 A = 1$

$\sin 2A = 2 \sin A \cos A$ $\quad \cos 2A = \cos^2 A - \sin^2 A$ $\quad \tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\quad \cos A \cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2}$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\quad \sin A \sin B = \dfrac{\cos(A-B) - \cos(A+B)}{2}$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ $\quad \sin A \cos B = \dfrac{\sin(A+B) + \sin(A-B)}{2}$

$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$ $\quad \cos^2 A = \dfrac{1 + \cos 2A}{2}$

$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$ $\quad \sin^2 A = \dfrac{1 - \cos 2A}{2}$

$\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$ $\quad \cos^3 A = \dfrac{3 \cos A + \cos 3A}{4}$

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$ $\quad \sin^3 A = \dfrac{3 \sin A - \sin 3A}{4}$

Relations between sides and angles of any plane triangle

In a plane triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a = b \cos C + c \cos B$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\text{area} = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s = \frac{1}{2}(a+b+c)$$

Relations between sides and angles of any spherical triangle

In a spherical triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$