

Maxima et minima.

T116

Finding maximum and minimum values

We now consider an example involving maxima and minima.

Example

Consider the function $f(x) = 4 \cos x + 3 \sin x - 3$. We might be interested in a question such as 'what are its maximum and minimum values?'

From the earlier work in this unit we can express the first two terms as a single trigonometric function:

$$\begin{aligned} f(x) &= 4 \cos x + 3 \sin x - 3 \\ &= 5 \cos(x - \alpha) - 3 \quad \text{where} \quad \tan \alpha = \frac{3}{4} \end{aligned}$$

Now the maximum value of the cosine function is 1 and this occurs when the angle $x - \alpha = 0$, i.e. when $x = \alpha$. So the maximum value of $f(x)$ must be $5 \times 1 - 3 = 2$.

What about the minimum value of $f(x)$? We know that the minimum value of the cosine function is -1 and this occurs when $x - \alpha = \pi$, i.e. when $x = \pi + \alpha$. So the minimum value of $f(x)$ is $5 \times -1 - 3 = -8$.

So, very quickly and with the minimum amount of work we have established a maximum value and a minimum value, and based upon this information we could go ahead and sketch a graph of $f(x)$. We see that the form $R \cos(x - \alpha)$ is a very powerful form for us to know how to use and for us to be able to formulate from expressions such as $4 \cos x + 3 \sin x$.

Exercises 3

For each of the following functions determine the maximum value and the smallest positive angle (in radians, to three decimal places) at which the maximum value occurs.

- a) $f(x) = 6 + 3 \cos x + 4 \sin x$ b) $f(x) = 3 - 4 \cos x + 3 \sin x$
c) $f(x) = 1 - 3 \cos x - 4 \sin x$ d) $f(x) = 2 + \cos x - \sin x$

Solutions

Exercises 1

- a) 13, 1.176 b) $\sqrt{10}$, 0.322 c) $\sqrt{10}$, -0.322 d) $\sqrt{61}$, 0.695
e) 13, 1.966 f) $\sqrt{17}$, -0.245 g) $\sqrt{13}$, -2.159 h) $\sqrt{10}$, 1.893
i) $\sqrt{2}$, $\frac{\pi}{4}$ j) $\sqrt{2}$, $-\frac{\pi}{4}$ k) $\sqrt{2}$, $\frac{3\pi}{4}$ l) $\sqrt{2}$, $-\frac{3\pi}{4}$

Exercises 2

- a) 1.571, $\left(\frac{\pi}{2}\right)$ or 5.640 b) 0.644 or 4.712 $\left(\frac{3\pi}{2}\right)$ c) 2.498 or 4.712 $\left(\frac{3\pi}{2}\right)$
d) 4.069 e) 2.214 f) 3.142 (π) or 0.927

Exercises 3

- a) Max value 11 at 0.927 b) Max value 8 at 2.498
c) Max value 6 at 4.069 d) Max value $2 + \sqrt{2}$ at $\frac{7\pi}{4}$