RADIAN MEASURE

Outcome:

1. *Define the radian.*
2. *Convert measurements in degrees to radians, and vice-versa.*
3. *Express angles in multiples of π radians.*
4. *Use the relationships $s = r\theta$ for the length of an arc and $A = \frac{1}{2}r^2\theta$ for the area of a sector.*
5. *Solve problems involving lengths of arcs, areas and angles in radians.*

RADIAN MEASURE

We have seen that an angle is usually measured in degrees but there is another way of measuring an angle. In this, the unit is known as the radian (abbreviation rad).

Referring to Fig. 12.1 gives

$$\text{Angle in radians} = \frac{\text{Length of arc}}{\text{Radius of circle}}$$

$$\theta \text{ radians} = \frac{l}{r}$$

$$l = r\theta$$

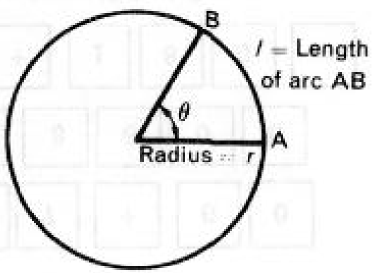

Fig. 12.1

Hence

$$\text{Length of arc} = r\theta$$

RELATION BETWEEN RADIANS AND DEGREES

If we make the arc AB equal to a semi-circle then

$$\text{Length of arc} = \pi r$$

and

$$\text{Angle in radians} = \frac{\pi r}{r} = \pi$$

Now the angle subtended by a semi-circle $= 180°$

Therefore $\pi \text{ radians} = 180°$

or $1 \text{ radian} = \frac{180°}{\pi} = 57.3°$

Thus to convert from degrees to radians

$$\boxed{\theta° = \frac{\pi\theta}{180} \text{ radians}}$$

Thus $30° = \frac{\pi(30)}{180} \text{ rad} = \frac{\pi}{6} \text{ rad}$

$90° = \frac{\pi}{2} \text{ rad}$ $\quad 180° = \pi \text{ rad}$

$45° = \frac{\pi}{4} \text{ rad}$ $\quad 270° = \frac{3\pi}{2} \text{ rad}$

$60° = \frac{\pi}{3} \text{ rad}$ $\quad 360° = 2\pi \text{ rad}$

To convert from radians to degrees

$$\boxed{\theta \text{ radians} = \left(\frac{180}{\pi} \times \theta\right)°}$$

DEGREES, MINUTES AND SECONDS

There are 60 seconds in 1 minute, or $60'' = 1'$

and 60 minutes in 1 degree, or $60' = 1°$

Thus 60×60 seconds in 1 degree, or $3600'' = 1°$

Modern calculating methods make the use of decimal degrees (e.g. 36.783°) more likely than the use of minutes and seconds.

EXAMPLE 12.1

Convert 29°37′29″ to radians stating the answer correct to 4 significant figures.

The first step is to convert the given angle into degrees and decimals of a degree.

$$29°37'29'' = 29 + \frac{37}{60} + \frac{29}{3600} = 29.625°$$

$$= \frac{\pi \times 29.625}{180} = 0.5171 \text{ radians}$$

Many scientific calculators will convert degrees, minutes and seconds into decimal degrees, and vice versa, using special keys—instructions for use of these keys will be given in the accompanying booklet.

However, the sequence given below will perform the calculation given in the solution shown above.

AC	2	9	Min	3	7	÷	6	0	
=	M+	2	9	÷	3	6	0	0	
=	M+	MR	×	π	÷	1	8	0	=

giving an answer of 0.5171 correct to 4 significant figures.

EXAMPLE 12.2

Convert 0.089 35 radians into degrees, minutes and seconds.

$$0.089\,35 \text{ radians} = \frac{0.089\,35 \times 180}{\pi} = 5.1194° = 5°7'10''$$

For calculators without decimal degree conversion facility the following sequence may be used—this is a reverse of the sequence used in the previous example.

AC	0	.	0	8	9	3	5
×	1	8	0	÷	π	=	

Display ①						
5.119 377 9	−	5	×	6	0	=

Display ②							Display ③
7.162 673 9	−	7	×	6	0	=	9.760 432 8

In this sequence it is necessary to record three results as they appear. The whole number, namely 5, in display ① is the number of degrees. The whole number 5 is then subtracted to leave the decimal part which is then multiplied by 60. The whole number, namely 7, in display ② is the number of minutes. The whole number, namely 7 is now subtracted to leave the decimal part which is then multiplied by 60 to give display ③—this figure is the number of seconds.

Thus the result is 5°7'10" to the nearest second.

COMPONENTS OF A CIRCLE

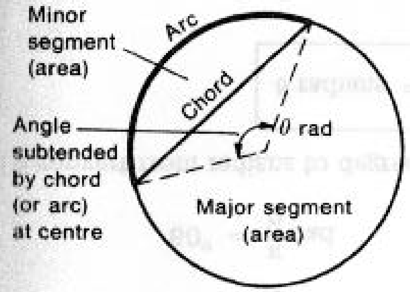

Fig. 12.2

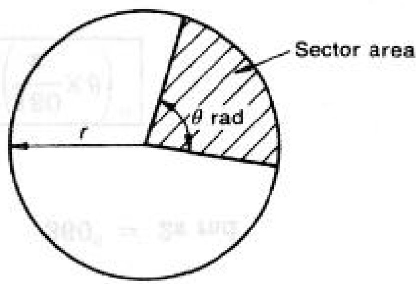

Fig. 12.3

AREA OF A SECTOR

The area of a circle = πr^2.

So, by proportion, (Fig. 12.2),

$$\text{Area of sector} = \pi r^2 \times \frac{\theta}{2\pi} = \tfrac{1}{2}r^2\theta$$

Summary

Length of arc = $r\theta$	θ in radians	or	$2\pi r\left(\frac{\theta°}{360}\right)$
Area of a sector = $\frac{1}{2}r^2\theta$	θ in radians	or	$\pi r^2\left(\frac{\theta°}{360}\right)$

EXAMPLE 12.3

Calculate a) the length of arc of a circle whose radius is 8 m and which subtends an angle of 56° at the centre, and b) the area of the sector so formed.

a) Length of arc $= 2\pi r \times \frac{\theta^\circ}{360} = 2 \times \pi \times 8 \times \frac{56}{360} = 7.82\text{ m}$

b) Area of sector $= \pi r^2 \times \frac{\theta^\circ}{360} = \pi \times 8^2 \times \frac{56}{360} = 31.28\text{ m}^2$

EXAMPLE 12.4

Find the angle of a sector of radius 35 mm and area 1020 mm²

Now Area of sector $= \frac{1}{2}r^2\theta$

and substituting the given values of

$$\text{Area} = 1020\text{ mm}^2 \quad \text{and} \quad r = 35\text{ mm}$$

we have $1020 = \frac{1}{2}(35)^2\theta$

from which $\theta = \frac{1020 \times 2}{35^2} = 1.67\text{ rad}$

$$= \frac{180 \times 1.67}{\pi} = 95.7^\circ$$

EXAMPLE 12.5

Water flows in a 400 mm diameter pipe to a depth of 300 mm. Calculate the wetted perimeter of the pipe and the area of cross-section of the water.

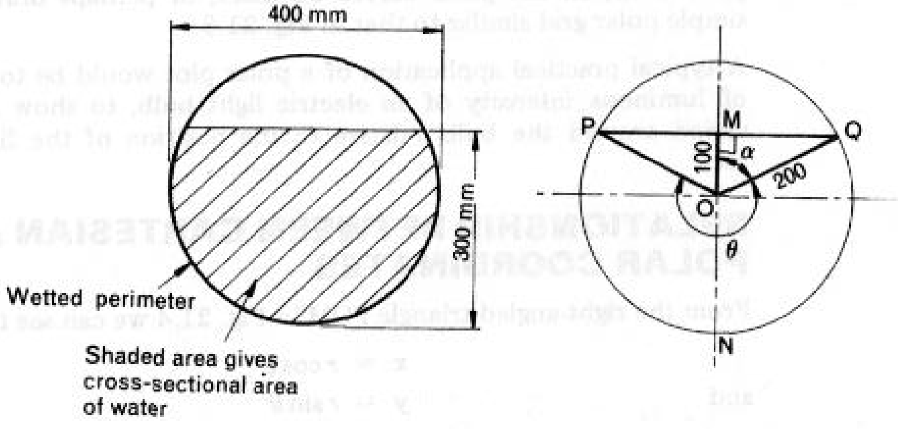

Fig. 12.4

From Fig. 12.4

the right-angled triangle MQO

$$\cos\alpha = \frac{OM}{OQ} = \frac{100}{200} = 0.5$$

$\therefore$ $\alpha = 60^\circ$

Also $\sin\alpha = \frac{MQ}{OQ}$

$\therefore$ $MQ = OQ\sin\alpha = 200\sin 60^\circ = 173.2\text{ mm}$

Now $\theta + 2\alpha = 360^\circ$

$\therefore$ $\theta = 360^\circ - 2(60^\circ) = 240^\circ$

Thus

$$\text{Wetted perimeter} = \text{Arc PNQ} = 2\pi r\left(\frac{\theta}{360}\right) = 2\pi(200)\left(\frac{240}{360}\right) = 838\text{ mm}$$

Also

$$\left(\begin{matrix}\text{Cross-sectional} \\ \text{area of water}\end{matrix}\right) = \left(\begin{matrix}\text{Area of} \\ \text{sector PNQ}\end{matrix}\right) + \left(\begin{matrix}\text{Area of} \\ \text{triangle POQ}\end{matrix}\right)$$

$$= \pi r^2\left(\frac{\theta}{360}\right) + \tfrac{1}{2}(PQ)(MO)$$

$$= \pi(200)^2\left(\frac{240}{360}\right) + \tfrac{1}{2}(2 \times 173.2)(100)$$

$$= 83\,780 + 17\,320$$

$$= 101\,000\text{ mm}^2$$

Exercise 12.1

1) Convert the following angles to radians stating the answers correct to 4 significant figures:

(a) 35° (b) 83°28′ (c) 19°17′32″ (d) 43°39′49″

2) Convert the following angles to degrees, minutes and seconds correct to the nearest second:

(a) 0.1732 radians (b) 1.5632 radians (c) 0.0783 radians

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