POLAR GRAPHS

- ...ine polar coordinates.
- ...e the relationship between polar and ...tesian co-ordinates.
- ...vert Cartesian to polar co-ordinates and ... versa.
4. *Plot graphs of functions defined in polar co-ordinates such as $r = a$, $\theta = \alpha$, $r = k\theta$, $r = \sin\theta$.*

POLAR COORDINATES

You have met previously Cartesian (or rectangular) co-ordinates with which P may be given as the point (x, y) as shown in Fig. 21.1.

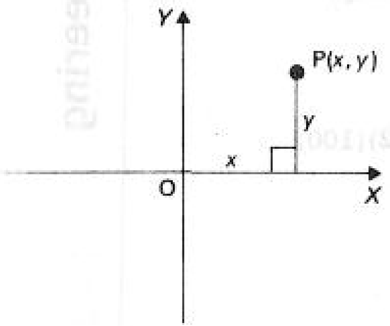


Fig. 21.1

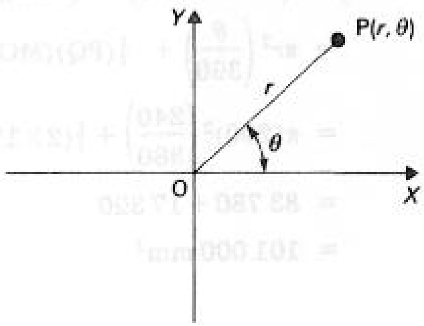


Fig. 21.2

Another way of giving the location of P is by using its distance r from the origin O, together with angle θ that OP makes with the horizontal axis OX (Fig. 21.2). Figure 21.3 shows five typical points, plotted on a polar grid, together with their respective polar co-ordinates.

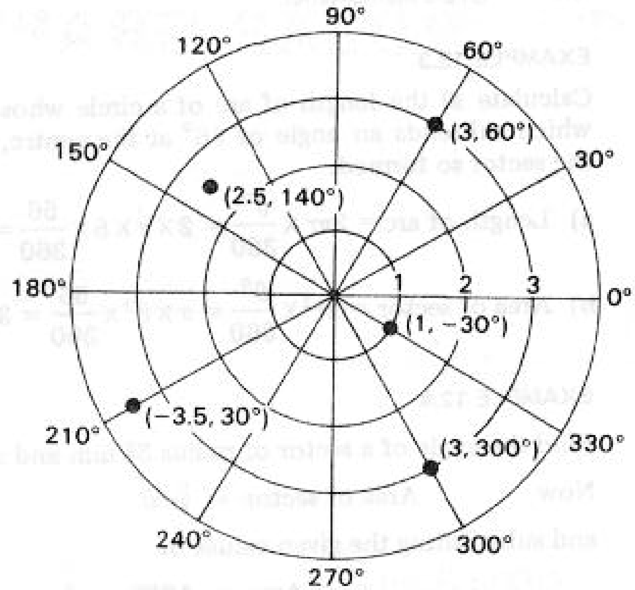


Fig. 21.3

Positive values of θ are always measured anticlockwise from OX whilst negative values are measured clockwise (see Fig. 21.3): you will not often meet the latter case. A negative value of r means that the 'radius length' is extended 'backwards' through O from the normal angle position: see point $(-3.5, 30°)$ in Fig. 21.3. Note that in polar co-ordinates it is possible to define a point in more than one way: for example the point $(3, 300°)$ in Fig. 21.3 could also be defined as $(-3, 120°)$.

Polar graph paper is available but is not so readily obtained as the common linear variety. However, it should be sufficient here for you to sketch the polar curves we meet, or perhaps draw your simple polar grid similar to that in Fig. 21.3.

A typical practical application of a polar plot would be to values of luminous intensity of an electric light bulb, to show how it varied around the bulb relative to the position of the filament etc.

RELATIONSHIP BETWEEN CARTESIAN AND POLAR COORDINATES

From the right-angled triangle POM in Fig. 21.4 we can see that:

$$x = r\cos\theta$$

and

$$y = r\sin\theta$$

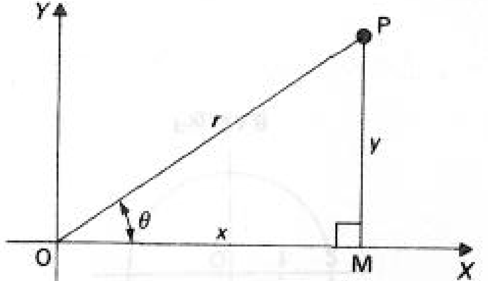


Fig. 21.4

Also $$r = \sqrt{x^2 + y^2}$$

and $$\tan\theta = \frac{y}{x}$$

Using the above relationships it is reasonably easy to convert from Cartesian to polar co-ordinates, and vice versa. Always make a sketch of the problem because this will enable you to see which quadrant you are dealing with.

EXAMPLE 21.1

Find the polar co-ordinates of the point $(-4, 3)$.

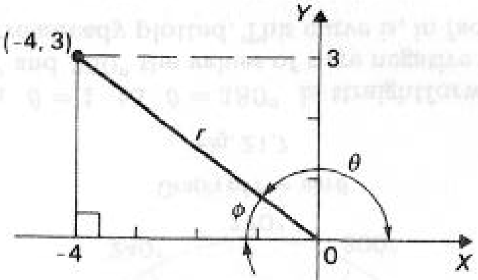


Fig. 21.5

From Fig. 21.5 $\tan\phi = \frac{3}{4} = 0.75$

$\therefore$ $\phi = 36.9°$

Thus $\theta = 180° - 36.9° = 143.1°$

Also $r = \sqrt{3^2 + 4^2} = 5$

Thus the point is $(5, 143.1°)$ in polar form.

EXAMPLE 21.2

Express in Cartesian form the point $(6, 231°)$.

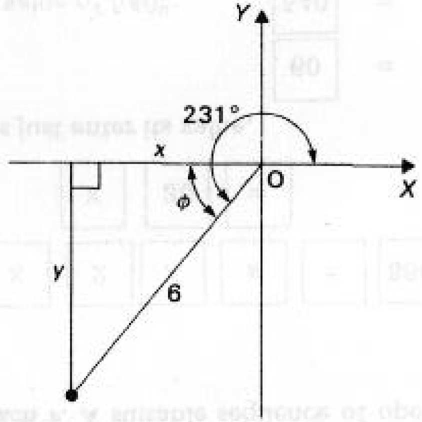


Fig. 21.6

From Fig. 21.6 $\phi = 231° - 180° = 51°$

Thus $x = 6\cos 51° = 3.78$

and $y = 6\sin 51° = 4.66$

Now, having drawn a diagram, we can see that both the x and y co-ordinates are negative.

Thus the Cartesian form of the point is $(-3.78, -4.66)$.

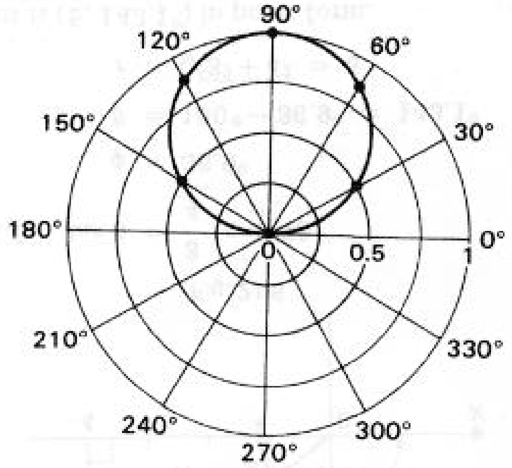
GRAPHS OF FUNCTIONS

When using Cartesian co-ordinates a graph may be drawn to illustrate y as a function of x. For example $y = mx + c$ represents a straight line graph. Similarly when using polar co-ordinates a graph may be drawn to illustrate r in terms of θ. In the examples which follow we will sketch some of the more common polar graphs and this should enable us to become familiar with their shapes.

EXAMPLE 21.3

Sketch the graph of $r = \sin\theta$ between $\theta = 0°$ and $\theta = 360°$.

From experience we know that $\sin\theta$ (and hence r) has a maximum value of $+1$, and a minimum value of -1. This will help initially in labelling our polar grid (Fig. 21.7). We may plot the graph from values obtained from our scientific calculator. We first measure off the angle θ and then measure off the length r along the angle boundary line.



Graph of $r = \sin\theta$

Fig. 21.7

Plotting from $\theta = 1$ to $\theta = 180°$ is straightforward. However, between 180° and 360° the values of r are negative and we merely repeat the curve already plotted. This curve is, in fact, a circle.

For interest you may like to verify that this is the only curve for *any* value of θ however large—both positive and negative.

EXAMPLE 21.4

Sketch the graph of $r = 2$.

Here no mention is made of the angle θ. This means that $r = 2$ defines the graph whatever value is given for θ. Thus the graph is a circle of radius 2 as shown in Fig. 21.8.

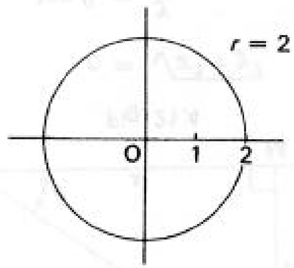


Fig. 21.8

EXAMPLE 21.5

Sketch the graph of $\theta = 40°$.

Here no mention is made of r. This means that $\theta = 40°$ defines the graph whatever value is given for r (whether positive or negative). Hence the graph is a straight line as shown in Fig. 21.9.

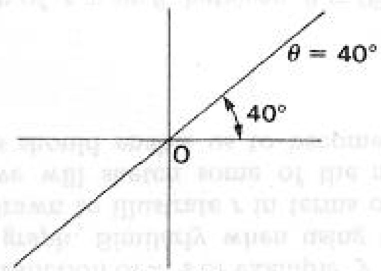


Fig. 21.9

EXAMPLE 21.6

Sketch the graph of $r = 3\theta$ for angle values equivalent to the range 0° to 540°.

As is usual in mathematics when an angle is used directly in calculations its value must be in radians. But we find it more convenient to plot the angle values using degrees and so we must be careful to find the corresponding angle values in radians when calculating r values.

Your scientific calculator may convert directly from degrees to radians. If not, we know that $360° = 2\pi$ rad or $1° = \frac{2\pi}{360}$ rad, and we may also use a constant multiplier facility to avoid entering this fraction for each r. A suitable sequence of operations would be:

For $\theta = 30°$:

[AC] [3] [×] [2] [×] [π] [÷] [360] [×]

[×] [30] [=] giving 1.57

and for other angles just enter its value,

e.g. for 60°: [60] = giving 3.14

and for the highest value of 540°: [540] = giving 28.27

The graph is known as an Archimedean spiral and is shown in Fig. 21.10.

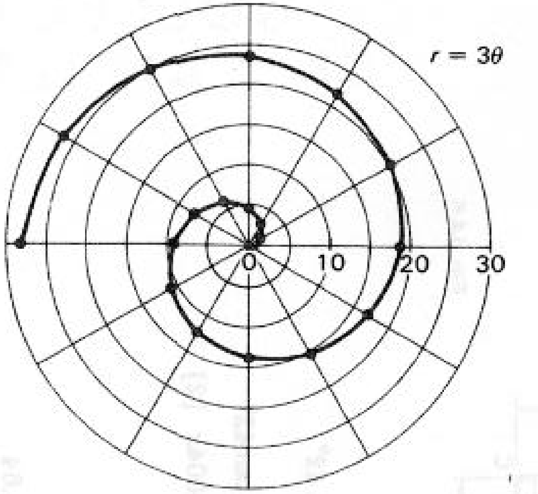


Fig. 21.10

Exercise 21.1

1) Find the polar co-ordinates of the following points:
(a) (5, 7) (b) (−2, 3) (c) (2, −3) (d) (−3, −4)

2) Find the Cartesian co-ordinates of the following points:
(a) (2, 35°) (b) (3, 127°) (c) (1.5, 240°)
(d) (0.6, 312°) (e) (2.3, −21°) (f) (−5, 130°)

Sketch the graphs for the polar equations in the questions which follow:

3) $r = 2\cos\theta$ 4) $r = 3$ 5) $\theta = 120°$

6) $r = \theta$ 7) $r = \sin^2\theta$ 8) $r = \cos^2\theta$

9) $r = 3\sin 2\theta$ 10) $r = \cos 3\theta$ 11) $r = a(1 + \cos\theta)$

[redacted]

The second fact is that there are also 180° in a triangle. So that if a right-angled triangle has two angles, say 90° and 55°, then the complementary angle is easily found as:

$\triangle < 180 - 90 - 55 = 35°$

Example 4.52

Calculate the length of *AB*, shown in Figure 4.23. Given that length *BC* = 200 mm.

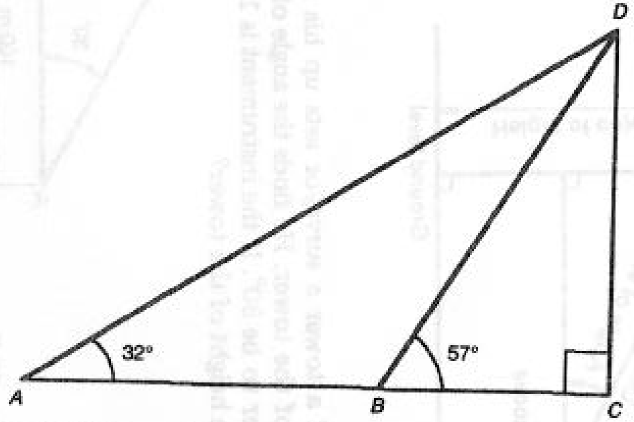


Figure 4.23 *Figure for Example 4.52*

To solve this problem, we need to remember the facts concerning complementary angles.

Then, $\angle ADC = 180 - 32 - 90 = 58°$ and with respect to the angle $CBD = 57°$, we can use the tangent ratio to find *CD*, then:

$\tan 57 = \frac{CD}{200}$ or $CD = 200 \tan 57$ and so $CD = (200)(1.5399)$

$CD = 308$ mm to the nearest mm

Now with respect to angle $ADC = 58°$, again using the tangent ratio, we get:

$\tan 58 = \frac{AC}{CD}$ or $AC = CD \tan 58 = (308)(1.6033) = 493$ mm, again to the nearest mm.

Then, required side $AB = AC - BC = 493 - 200 = 293$ mm.

In the above example we could go on to find the remaining sides of both triangles and so solve them completely.

[redacted]

ANGLE OF ELEVATION

XII

If you look upwards at an object the angle formed between the horizontal and your line of sight is called the *angle of elevation* (Fig. 14.3).

Climb

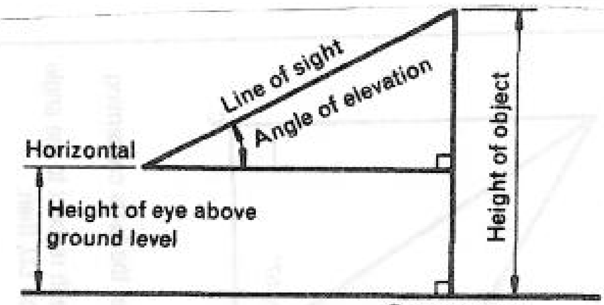


Fig. 14.3

EXAMPLE 14.3

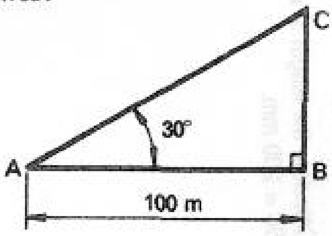
To find the height of a tower a surveyor sets up his theodolite 100 m from the base of the tower. He finds the angle of elevation of the top of the tower to be 30°. If the instrument is 1.5 m from the ground, what is the height of the tower?

In Fig. 14.4,

$$\frac{BC}{AB} = \tan 30^\circ$$

$$\therefore \quad BC = AB \tan 30^\circ$$

$$= 100 \tan 30^\circ = 57.7$$



Hence, height of tower = 57.7 + 1.5 = 59.2 m.

Fig. 14.4

EXAMPLE 14.4

To find the height of a pylon, a surveyor sets up a theodolite some distance from the base of the pylon and finds that the angle of elevation to the top of the pylon to be 30°. He then moves 60 m nearer to the pylon and finds that the angle of elevation is 42°. Find the height of the pylon assuming that the ground is horizontal and that the theodolite stands 1.5 m above the ground.

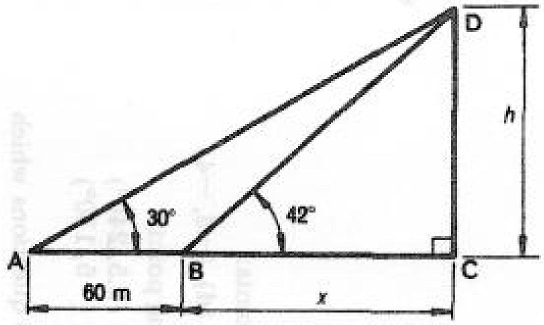


Fig. 14.5

Referring to Fig. 14.5, let $BC = x$ and $DC = h$.

In $\triangle ACD$,

$$\frac{DC}{AC} = \tan 30^\circ$$

$$\therefore \quad DC = AC \tan 30^\circ$$

or $\quad h = 0.5774(x + 60) \quad [1]$

In $\triangle BDC$,

$$\frac{DC}{BC} = \tan 42^\circ$$

$$\therefore \quad DC = BC \tan 42^\circ$$

or $\quad h = 0.9004x \quad [2]$

From equation [2], $x = \dfrac{h}{0.9004} = 1.1106h$

Substituting for x in equation [1] gives

$$h = 0.5774(1.1106h + 60) = 0.6413h + 34.64$$

$$h - 0.6413h = 34.64$$

or $\quad 0.3587h = 34.64$

from which $\quad h = \dfrac{34.64}{0.3587} = 96.6\text{ m}$

Hence the height of the pylon is 96.6 + 1.5 = 98.1 m

XIII

Decent

ANGLE OF DEPRESSION

If you look down at an object, the angle formed between the horizontal and your line of sight is called the *angle of depression* (Fig. 14.6).

EXAMPLE 14.5

From the top floor window of a house, 14 m above ground level, the angle of depression of an object in the street is 52°. How far is the object from the house?

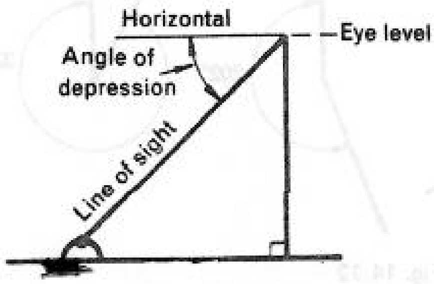


Fig. 14.6

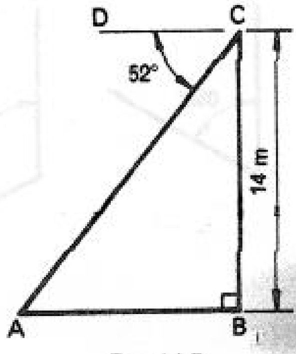


Fig. 14.7

The conditions are shown in Fig. 14.7. Since the angle of depression is 52°, $\angle ACD = 52°$.

$$\angle ACB = 90° - 52° = 38°$$

In $\triangle ABC$, $\dfrac{AB}{CB} = \tan 38°$

$\therefore \quad AB = CB \tan 38° = 14 \tan 38° = 10.9$

Hence the object is 10.9 m from the house.

BEARINGS

The four cardinal directions are North, South, East and West (Fig. 14.8). The directions NE, NW, SE and SW are frequently used and are as shown in the diagram. A bearing of N20°E means an angle 20° measured from N towards E as shown in Fig. 14.9. Similarly a bearing of S40°E means an angle of 40° measured from S towards E (Fig. 14.10). A bearing of N50°W means an angle of 50° measured from N towards W (Fig. 14.11).

Bearings quoted in this way are always measured from N and S and never from E and W.

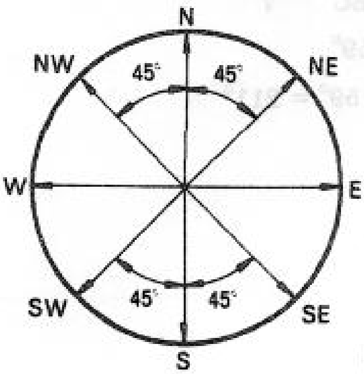


Fig. 14.8

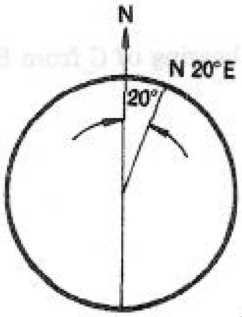


Fig. 14.9

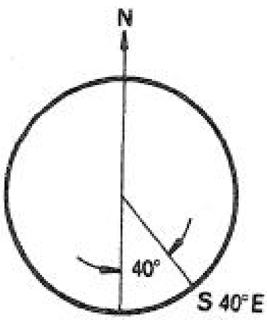


Fig. 14.10

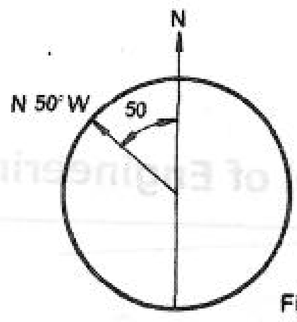


Fig. 14.11

XIV

Another way of stating bearings is to measure the angle from N in a clockwise direction, N being taken as 0°. Three figures are always stated. For example 005° is written instead of 5° and 035° instead of 35° and so on. E will be 090°, S 180° and W 270°. Some typical bearings are shown in Fig. 14.12.

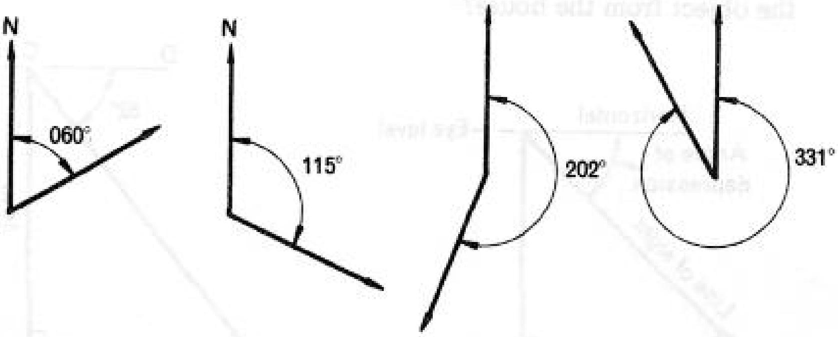


Fig. 14.12

EXAMPLE 14.6

In making a survey it is found that B is a point due east of a point A and a point C is 6 km due south of A. The distance BC is 7 km. Calculate the bearing of C from B.

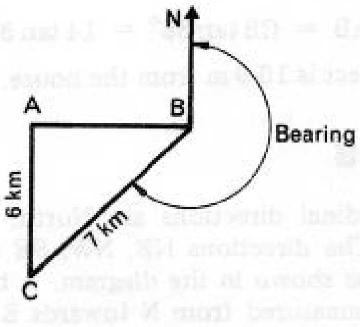


Fig. 14.13

In right-angled ΔABC, Fig. 14.13 we have

$$\sin \angle B = \frac{AC}{BC} = \frac{6}{7}$$

$$\therefore \qquad \angle B = 59^\circ$$

The bearing of C from B = 270° − 59° = 211°