Logarithmic differentiation

5.1 Introduction

The technique of **logarithmic differentiation** is useful when we need to differentiate a cumbersome product. The method involves taking *the natural logarithm of the function to be differentiated.*

5.2 Logarithmic differentiation

We gather together some important results that are commonly used in logarithmic differentiation.

1. $\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$
2. $\frac{d}{dx}(\ln f(x)) = \frac{1}{f}\frac{df}{dx}$
3. $\ln(e^x) = x$
4. $\ln(e^{f(x)}) = f(x)$
5. $\ln AB = \ln A + \ln B$
6. $\ln \frac{A}{B} = \ln A - \ln B$
7. $\ln A^n = n \ln A$

The method of logarithmic differentiation is illustrated in the following examples.

Example 5.1

Given $y = t^2(1+t)^3$ find $\frac{dy}{dt}$.

Solution

We could use the product rule to find $\frac{dy}{dt}$. However, we shall use logarithmic differentiation to illustrate the technique. Taking the natural logarithm of both sides of the given equation and applying the laws of logarithms yields

$$\begin{aligned} \ln y &= \ln(t^2(1+t)^3) \\ &= \ln t^2 + \ln(1+t)^3 \\ &= 2\ln t + 3\ln(1+t) \end{aligned}$$

Now, both sides are differentiated w.r.t. t.

$$\frac{d}{dt}(\ln y) = \frac{d}{dt}(2\ln t) + \frac{d}{dt}(3\ln(1+t))$$

$$\frac{1}{y}\frac{dy}{dt} = 2\left(\frac{1}{t}\right) + 3\left(\frac{1}{1+t}\right)$$

$$\frac{dy}{dt} = y\left(\frac{2}{t} + \frac{3}{1+t}\right)$$

$$= t^2(1+t)^3\left(\frac{2}{t} + \frac{3}{1+t}\right)$$

Example 5.2

Given that

$$y = (t+1)^7(2t+3)^4(2t-1)^5$$

find $\frac{dy}{dt}$.

Solution

Taking the natural logarithm of both sides and applying the laws of logarithms yields

$$\ln y = \ln((t+1)^7(2t+3)^4(2t-1)^5)$$

$$= \ln(t+1)^7 + \ln(2t+3)^4 + \ln(2t-1)^5$$

$$= 7\ln(t+1) + 4\ln(2t+3) + 5\ln(2t-1)$$

We now differentiate both sides w.r.t. t.

$$\frac{d}{dt}(\ln y) = \frac{d}{dt}(7\ln(t+1)) + \frac{d}{dt}(4\ln(2t+3)) + \frac{d}{dt}(5\ln(2t-1))$$

$$\frac{1}{y}\frac{dy}{dt} = \frac{7}{t+1} + \frac{8}{2t+3} + \frac{10}{2t-1}$$

$$\frac{dy}{dt} = y\left(\frac{7}{t+1} + \frac{8}{2t+3} + \frac{10}{2t-1}\right)$$

$$\frac{dy}{dt} = (t+1)^7(2t+3)^4(2t-1)^5\left(\frac{7}{t+1} + \frac{8}{2t+3} + \frac{10}{2t-1}\right)$$

Example 5.3

Find $\frac{dy}{dx}$ given $y = e^{2x}x^3(1-x)^4$.

Solution

Taking the natural logarithm of both sides and applying the laws of logarithms gives

$$\ln y = \ln(e^{2x}x^3(1-x)^4)$$

$$= \ln e^{2x} + \ln x^3 + \ln(1-x)^4$$

$$= 2x + 3\ln x + 4\ln(1-x$$

$2x$ $\ln x$, $4\ln(1-x)$

Differentiating w.r.t. x gives

and so

$$\frac{dy}{dx} = y\left(2 + \frac{3}{x} - \frac{4}{1-x}\right)$$

$$\frac{dy}{dx} = e^{2x}x^3(1-x)^4\left(2 + \frac{3}{x} - \frac{4}{1-x}\right)$$

Example 5.4

Find $\frac{dz}{dt}$ given $z(t) = \sqrt{1+t^2}\cos^4 t$.

Solution

Taking the natural logarithm of both sides gives

$$\ln z = \ln(\sqrt{1+t^2}\cos^4 t)$$

$$= \ln\sqrt{1+t^2} + \ln\cos^4 t$$

$$= \qquad + \qquad$$

$$\frac{1}{2}\ln(1+t^2) \quad + \quad 4\ln\cos t$$

Differentiating both sides w.r.t. t gives

$$\frac{1}{z}\frac{dz}{dt} =$$

$$\frac{t}{1+t^2} - \frac{4\sin t}{\cos t}$$

So

$$\frac{dz}{dt} = z\left(\frac{t}{1+t^2} - \frac{4\sin t}{\cos t}\right)$$

$$= \sqrt{1+t^2}\cos^4 t\left(\frac{t}{1+t^2} - 4\tan t\right)$$

Exercises

1 Find $\frac{dy}{dx}$ where y is given by

(a) $x^5(3x+7)^9$ (b) $x^8(5x-1)^6$

(c) $(3x+2)^4(9x-5)^7$

(d) $(1-3x)^4(2-7x)^6$

(e) $(5+2x)^4(5x+2)^3$

2 Find the derivative of each of the following functions:

(a) $z(t) = e^{3t}(2t-5)^3(3t+1)^4$

(b) $h(t) = 3e^{-6t}t^7(t+6)^3$

(c) $M(p) = -p^4\sin^5 p$

3 Find the rate of change of

$$q(t) = 2e^{-t/2}\cos 2t$$

when $t = 1$.

4 Find the derivative of the following functions using logarithmic differentiation:

(a) $a(t) = (1+t^2)^3(1-t^2)^4\sin^3 t$

(b) $b(r) = e^{-r}\sin^5 2r\cos^4 3r$

(c) $K(p) = 6p\sqrt{\sin p}(\cos 2p)^{1/3}$

(d) $N(y) = \frac{(1+\sqrt{y})^4(6+7y)^3}{}$

5) FUNCTION OF A FUNCTION; rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$y = (2x^3 - 5x)^5$

(parametric differentiation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy/du}{dx/du}$$

Differentiation rules; $y = ax^n \rightarrow \frac{dy}{dx} = anx^{n-1}$

for $\frac{dy}{du}(2x^3 - 5x)^5 = 5(2x^3 - 5x)^{5-1} = \underline{5(2x^3 - 5x)^4}$

$\frac{dx}{du}(2x^3 - 5x) = (3)(2)x^{3-1} - (1)(5)x^{1-1}$

$= \underline{6x^2 - 5}$

$\therefore$ ANSWER $\frac{dy}{dx} = \underline{\underline{5(2x^3 - 5x)^4 (6x^2 - 5)}}$

6) SUCCESSIVE DIFFERENTIATION:-

If $y = 3x^4 + 2x^3 - 3x + 2$ find 6.1) $\frac{dy}{dx}$ 6.2) $\frac{d^2y}{dx^2}$

6.1) $\frac{dy}{dx}(3x^4 + 2x^3 - 3x + 2) = ((4)(3)x^{4-1} + (3)(2)x^{3-1} - (1)(3)x^{1-1} + \cancel{2})$

ANSWER $= \underline{\underline{12x^3 + 6x^2 - 3}}$

6.2) $\frac{d^2y}{dx^2}(12x^3 + 6x^2 - 3) = ((3)(12)x^{3-1} + (2)(6)x^{2-1} - \cancel{3})$

ANSWER $= \underline{\underline{36x^2 + 12x}}$

7) LOGARITHMIC DIFFERENTIATION;

$$y = \frac{(x-2)(x+1)}{(x-1)(x+3)}$$

$$\ln y = \ln\left(\frac{(x-2)(x+1)}{(x-1)(x+3)}\right) \leftarrow$$ Take natural Logs of both sides!

p.t.o..

7/CONTINUED:-

$$\therefore \ln y = \frac{\ln((x-2)(x+1))}{\ln((x-1)(x+3))}$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\ln\frac{(x-2)(x+1)}{(x-1)(x+3)}$$

rules:

$\ln A^n \rightarrow n \ln A$

$$\frac{d}{dx}\ln(y) = \frac{1}{y}\frac{dy}{dx}$$

USING THE QUOTIENT RULE:

$$\frac{1}{y}\frac{dy}{dx} = \frac{\left(\frac{1}{x-2}\right)\left(\frac{1}{x+1}\right)}{(x-1)(x+3)} = \frac{(x-2)(x+1)}{1(x-1)\,1(x+3)}$$

$(x-2)(x+1)$

$x^2 - 2x + 1x - (2)(1)$

$= x^2 - x - 2$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$(x-1)(x+3)$

$x^2 - 1x + 3x - 1(3)$

$x^2 + 2x - 3$

$v = (x-1)(x+3)$

$u = \left(\frac{1}{x-2}\right)\left(\frac{1}{x+1}\right)$

$\frac{du}{dx} = (2)x^{2-1} + (1)1x^{1-1} \cancel{-2}^{\,0} = 2x - 1$

$\frac{dv}{dx} = (2)x^{2-1} + (1)2x^{1-1} \cancel{+3}^{\,0} = 2x + 2$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{(x-1)(x+3)\ 2x-1 - ((x-2)(x+1)\ 2x+2)}{(x-1)^2(x+3)^2}$$

$$= \frac{(x^2+2x-3)(2x-1) - (x^2-x-2)(2x+2)}{(x-1)^2(x+3)^2}$$

$$= \frac{(2x^3 + 4x^2 - 6x - x^2 - 2x + 3) - (2x^3 - 2x^2 - 4x + 2x^2 - 2x - 4)}{(x-1)^2(x+3)^2}$$

$$= \frac{(\cancel{2x^3}^{\,0} + 4x^2 + 6x - x^2 - 2x + 3) - (\cancel{2x^3}^{\,0} - \cancel{2x^2}^{\,0} + 4x + \cancel{2x^2}^{\,0} + 2x - 4)}{(x-1)^2(x+3)^2}$$

$2x^3$ terms cancel out

$2x^2$ terms cancel out

$4x^2(-x^2) = 3x^2$, $6x - 2x - (4x+2x) = 6x - 2x - 6x = 2x$, $3 - (-4) = -7$

$$= \frac{3x^2 - 2x - 7}{(x-1)^2(x+3)^2}$$

ANSWER