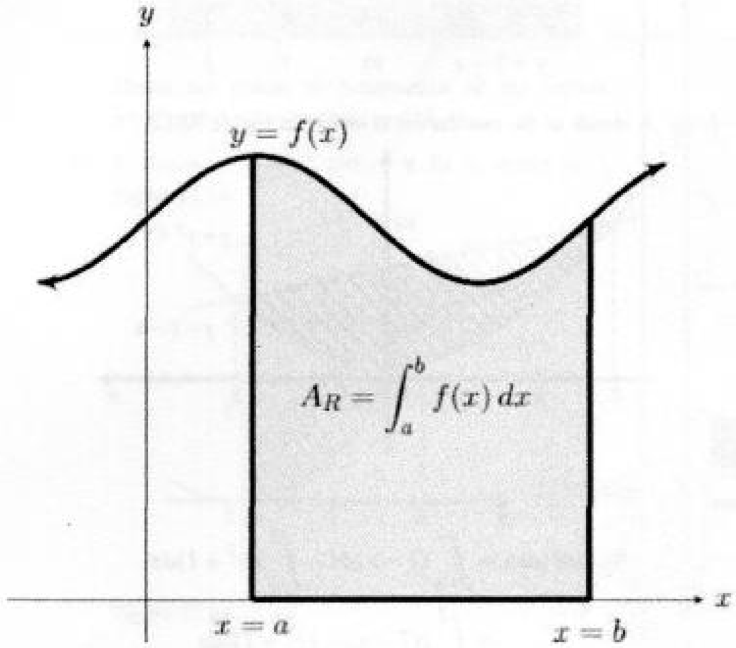
Integral Areas

Integral area problems. Week 25



This [redacted] Section deals with a specific application of integrals to find the area under simple curves, area between lines and arcs of circles, parabolas and ellipses, and finding the area bounded by the above said curves.

8.1.1 The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by the formula:

$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

8.1.2 The area of the region bounded by the curve $x = \phi(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula:

$$\text{Area} = \int_c^d x\,dy = \int_c^d \phi(y)\,dy$$

8.1.3 The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by the formula.

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b]$$

8.1.4 If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then

$$\text{Area} = \int_a^c [f(x) - g(x)]\,dx + \int_c^b (g(x) - f(x))\,dx$$

8.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the area of the curve $y = \sin x$ between 0 and π.

Solution We have

$$\text{Area OAB} = \int_0^\pi y\,dx = \int_0^\pi \sin x\,dx = |-\cos x|_0^\pi$$

$$= \cos 0 - \cos \pi = 2 \text{ sq units.}$$

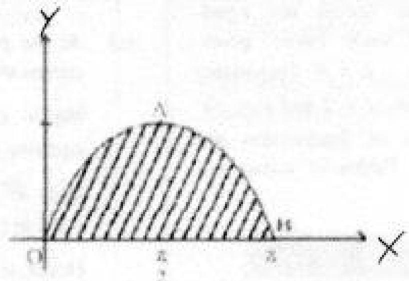


Fig. 8.1

Method,

72.4 The area between curves

The area enclosed between curves $y = f_1(x)$ and $y = f_2(x)$ (shown shaded in Figure 72.11) is given by:

$$\text{shaded area} = \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \int_a^b [f_2(x) - f_1(x)]\,dx$$

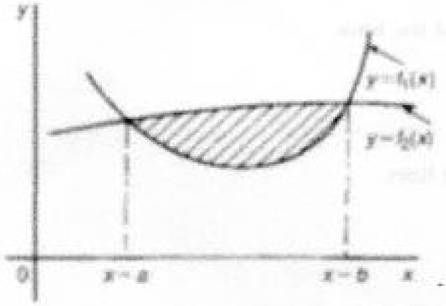


Figure 72.11

Problem 11. Determine the area enclosed between the curves $y=x^2+1$ and $y=7-x$

At the points of intersection, the curves are equal. Thus, equating the y-values of each curve gives: $x^2 + 1 = 7 - x$, from which $x^2 + x - 6 = 0$. Factorising gives $(x - 2)(x + 3) = 0$, from which, $x = 2$ and $x = -3$. By firstly determining the points of intersection the range of x-values has been found. Tables of values are produced as shown below.

x	−3	−2	−1	0	1	2
$y = x^2 + 1$	10	5	2	1	2	5

x	−3	0	2
$y = 7 - x$	10	7	5

A sketch of the two curves is shown in Figure 72.12.

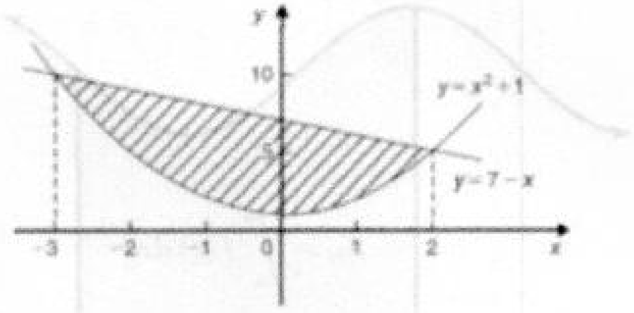


Figure 72.12

$$\text{Shaded area} = \int_{-3}^{2} (7 - x)\,dx - \int_{-3}^{2} (x^2 + 1)\,dx$$

$$= \int_{-3}^{2} [(7 - x) - (x^2 + 1)]\,dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right)$$

$$= \mathbf{20\frac{5}{6}} \textbf{ square units}$$

Problem 12. (a) Determine the coordinates of the points of intersection of the curves $y = x^2$ and $y^2 = 8x$. (b) Sketch the curves $y = x^2$ and $y^2 = 8x$ on the same axes. (c) Calculate the area enclosed by the two curves.

(a) At the points of intersection the coordinates of the curves are equal. When $y = x^2$ then $y^2 = x^4$

Hence at the points of intersection $x^4 = 8x$, by equating the y^2 values.

Thus $x^4 - 8x = 0$, from which $x(x^3 - 8) = 0$, i.e. $x = 0$ or $(x^3 - 8) = 0$

Hence at the points of intersection $x = 0$ or $x = 2$

When $x = 0$, $y = 0$ and when $x = 2$, $y = 2^2 = 4$

Hence the points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are (0, 0) and (2, 4)

(b) A sketch of $y = x^2$ and $y^2 = 8x$ is shown in Figure 72.13

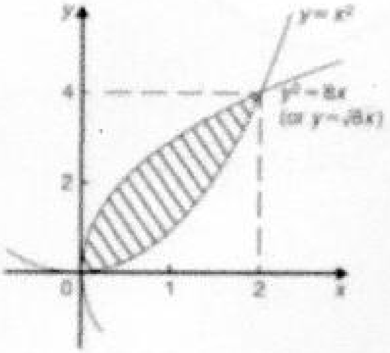


Figure 72.13

(c) **Shaded area** $= \int_0^2 (\sqrt{8x} - x^2)\,dx$

$$= \int_0^2 \{(\sqrt{8})x^{1/2} - x^2\}\,dx$$

$$= \left[(\sqrt{8})\frac{x^{3/2}}{\left(\frac{3}{2}\right)} - \frac{x^3}{3}\right]_0^2$$

$$= \left\{\frac{\sqrt{8}\sqrt{8}}{\left(\frac{3}{2}\right)} - \frac{8}{3}\right\} - \{0\}$$

$$= \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$$

$$= 2\frac{2}{3} \text{ square units}$$

Problem 13. Determine by integration the area bounded by the three straight lines $y = 4 - x$, $y = 3x$ and $3y = x$

Each of the straight lines is shown sketched in Figure 72.14.

Shaded area $= \int_0^1 \left(3x - \frac{x}{3}\right)dx + \int_1^3 \left[(4 - x) - \frac{x}{3}\right]dx$

Intersection points P1 & P2:

$y = 3x$, $y = 4 - x$ P1 → $3x = 4 - x$

$\therefore 0 = 4x - 4$

$\therefore x = \frac{4}{4} = 1$ ← $y = 3(1)$ $\therefore y = 3$

$3y = x$, $y = 4 - x$ P2 → $\frac{x}{3} = 4 - x$

$\therefore 0 = 1\frac{1}{3}x - 4$

$\therefore x = \frac{4}{1\frac{1}{3}} = 3$ ← $y = \frac{x}{3}$ when $x = 3$ $\therefore y = 1$

OR $y = 4 - x \therefore 4 - 3 = 1$

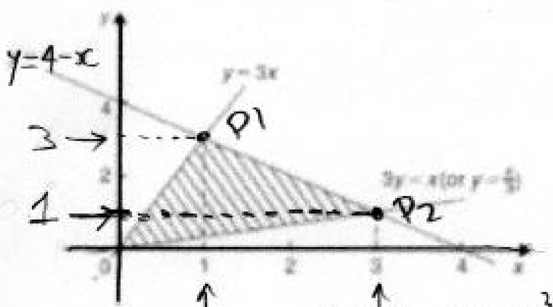


Figure 72.14

$$= \left[\frac{3x^{1+1}}{1+1} - \frac{1}{3}\frac{x^{1+1}}{1+1}\right]_0^1 + \left[4x - \frac{x^{1+1}}{1+1} - \frac{x^{1+1}}{3\times 2}\right]_1^3$$

RULES:– $k \to kx + c$, $x^a \to \frac{x^{a+1}}{a+1}$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$\frac{1}{3} \cdot \frac{x^2}{2} = \frac{x^2}{6}$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right]$$

$$+ \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right) - \left(4 - \frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right)$$

$$= 4 \text{ square units}$$

Now try the following Practice Exercise :–

1. Determine the coordinates of the points of intersection and the area enclosed between the parabolas $y^2 = 3x$ and $x^2 = 3y$
2. Sketch the curves $y = x^2 + 3$ and $y = 7 - 3x$ and determine the area enclosed by them
3. Determine the area enclosed by the curves $y = \sin x$ and $y = \cos x$ and the y-axis
4. Determine the area enclosed by the three straight lines $y = 3x$, $2y = x$ and $y + 2x = 5$